

CUBIC FOURFOLDS CONTAINING A PLANE AND A QUINTIC DEL PEZZO SURFACE

ASHER AUEL, MARCELLO BERNARDARA, MICHELE BOLOGNESI,
AND ANTHONY VÁRILLY-ALVARADO

ABSTRACT. We isolate a class of smooth rational cubic fourfolds X containing a plane whose associated quadric surface bundle does not have a rational section. This is equivalent to the nontriviality of the Brauer class β of the even Clifford algebra over the K3 surface S of degree 2 associated to X . Specifically, we show that in the moduli space of cubic fourfolds, the intersection of divisors $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has five irreducible components. In the component corresponding to the existence of a tangent conic, we prove that the general member is both pfaffian and has β nontrivial. Such cubic fourfolds provide twisted derived equivalences between K3 surfaces of degree 2 and 14, hence further corroboration of Kuznetsov's derived categorical conjecture on the rationality of cubic fourfolds.

INTRODUCTION

Let X be a *cubic fourfold*, i.e., a smooth cubic hypersurface $X \subset \mathbb{P}^5$ over the complex numbers. Determining the rationality of X is a classical question in algebraic geometry. Some classes of rational cubic fourfolds have been described by Fano [8], Tregub [29], [30], and Beauville–Donagi [4]. In particular, *pfaffian* cubic fourfolds, defined by pfaffians of skew-symmetric 6×6 matrices of linear forms, are rational. Equivalently, a cubic fourfold is pfaffian if and only if it contains a quintic del Pezzo surface, see [3, Prop. 9.1(a)]. Hassett [10] describes, via lattice theory, divisors \mathcal{C}_d in the moduli space \mathcal{C} of cubic fourfolds. In particular, \mathcal{C}_{14} is the closure of the locus of pfaffian cubic fourfolds and \mathcal{C}_8 is the locus of cubic fourfolds containing a plane. Hassett [11] identifies countably many divisors of \mathcal{C}_8 consisting of rational cubic fourfolds with trivial *Clifford invariant*. Nevertheless, it is expected that the general cubic fourfold (and the general cubic fourfold containing a plane) is nonrational. At present, however, not a single cubic fourfold is provably nonrational.

In this work, we study rational cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ with nontrivial Clifford invariant, hence not contained in the divisors of \mathcal{C}_8 described by Hassett. Let $A(X)$ be the lattice of algebraic 2-cycles on X up to rational equivalence and d_X the discriminant of the intersection form on $A(X)$. Our main result is a complete description of the irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$.

Theorem A. *There are five irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, indexed by the discriminant $d_X \in \{21, 29, 32, 36, 37\}$ of a general member. The Clifford invariant of a general cubic fourfold X in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ is trivial if and only if d_X is odd. The pfaffian locus is dense in the $d_X = 32$ component.*

In particular, the general cubic fourfold in the $d_X = 32$ component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ is rational and has nontrivial Clifford invariant, thus answering a question of Hassett [11, Rem. 4.3]. We also provide a geometric description of this component: its general member has a *tangent conic* to the sextic degeneration curve of the associated quadric surface bundle (see Proposition 8).

At the same time, we also answer a question of E. Macrì and P. Stellari, as these cubic fourfolds also provide a nontrivial corroboration of Kuznetsov's derived categorical conjecture on the rationality of cubic fourfolds containing a plane.

Kuznetsov [21] establishes a semiorthogonal decomposition of the bounded derived category

$$D^b(X) = \langle \mathbf{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

The category \mathbf{A}_X has the remarkable property of being a 2-Calabi–Yau category, essentially a noncommutative deformation of the derived category of a K3 surface. Based on evidence from

2010 *Mathematics Subject Classification.* 11E20, 11E88, 14C30, 14F05, 14E08, 14F22, 14J28, 15A66.

Key words and phrases. Cubic fourfold, rationality, quadric bundle, Clifford algebra, derived category.

known cases as well as general categorical considerations, Kuznetsov conjectures that the category \mathbf{A}_X contains all the information about the rationality of X .

Conjecture (Kuznetsov). *A complex cubic fourfold X is rational if and only if there exists a K3 surface S and an equivalence $\mathbf{A}_X \cong \mathbf{D}^b(S)$.*

If X contains a plane, a further geometric description of \mathbf{A}_X is available. Indeed, X is birational to the total space of a quadric surface bundle $\tilde{X} \rightarrow \mathbb{P}^2$ by projecting from the plane. We assume that the degeneration divisor is a smooth sextic curve $D \subset \mathbb{P}^2$. The discriminant double cover $S \rightarrow \mathbb{P}^2$ branched along D is then a K3 surface of degree 2 and the even Clifford algebra gives rise to a Brauer class $\beta \in \text{Br}(S)$, called the *Clifford invariant* of X . Via mutations, Kuznetsov [21, Thm. 4.3] establishes an equivalence $\mathbf{A}_X \cong \mathbf{D}^b(S, \beta)$ with the bounded derived category of β -twisted sheaves on S .

By classical results in the theory of quadratic forms (see [1, Thm. 2.24]), β is trivial if and only if the quadric surface bundle $\tilde{X} \rightarrow \mathbb{P}^2$ has a rational section. In particular, if $\beta \in \text{Br}(S)$ is trivial then X is rational and Kuznetsov’s conjecture is verified. This should be understood as the *trivial case* of Kuznetsov’s conjecture for cubic fourfolds containing a plane.

Conjecture (Kuznetsov “containing a plane”). *Let X be a smooth complex cubic fourfold containing a plane, S the associated K3 surface of degree 2, and $\beta \in \text{Br}(S)$ the Clifford invariant. Then X is rational if and only if there exists a K3 surface S' and an equivalence $\mathbf{D}^b(S, \beta) \cong \mathbf{D}^b(S')$.*

To date, this variant of Kuznetsov’s conjecture is only known to hold in the trivial case (where β is trivial and $S = S'$). E. Macrì and P. Stellari asked if there was class of smooth rational cubic fourfolds containing a plane that verify this variant of Kuznetsov’s conjecture in a nontrivial way, i.e., where β is not trivial and there exists a different K3 surface S' and an equivalence $\mathbf{D}^b(S, \beta) \cong \mathbf{D}^b(S')$. The existence of such fourfolds is not *a priori* clear: while a general cubic fourfold containing a plane has nontrivial Clifford invariant, the existence of *rational* such fourfolds was only intimated in the literature.

Theorem B. *Let X be a general member of the $d_X = 32$ component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, i.e., a smooth pfaffian cubic fourfold X containing a plane with nontrivial Clifford invariant $\beta \in \text{Br}(S)$. There exists a K3 surface S' of degree 14 and a nontrivial twisted derived equivalence $\mathbf{D}^b(S, \beta) \cong \mathbf{D}^b(S')$.*

The outline of this paper is as follows. In §1, we study Hodge theoretic and geometric conditions for the nontriviality of the Clifford invariant (see Propositions 3 and 4). In §2, we analyze the irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, proving the first two statements of Theorem A. We also answer a question of F. Charles on cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ with trivial Clifford invariant (see Theorem 6 and Proposition 7). Throughout, we use the work of Looijenga [23], Laza [22], and Mayanskiy [25] on the realizability of lattices of algebraic cycles on a cubic fourfold. In §3, we recall some elements of the theory of homological projective duality and prove Theorem B. Finally, in §4, we prove the final statement of Theorem A, that the pfaffian locus is dense in the $d_X = 32$ component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, by expliciting a single point in the intersection. For the verification, we are aided by Magma [6], adapting some of the computational techniques developed in [13].

Throughout, we are guided by Hassett [11, Rem. 4.3], who suggests that rational cubic fourfolds containing a plane with nontrivial Clifford invariant ought to lie in $\mathcal{C}_8 \cap \mathcal{C}_{14}$. While the locus of pfaffian cubic fourfolds is dense in \mathcal{C}_{14} , it is not true that the locus of pfaffians containing a plane is dense in $\mathcal{C}_8 \cap \mathcal{C}_{14}$. In Theorem A, we find a suitable component affirming Hassett’s suggestion.

Acknowledgments. Much of this work has been developed during visits of the authors at the Max Planck Institut für Mathematik in Bonn, Universität Duisburg–Essen, Université Rennes 1, ETH Zürich, and Rice University. The hospitality of each institute is warmly acknowledged. The first and fourth authors are partially supported by NSF grant MSPRF DMS-0903039 and DMS-1103659, respectively. The second author was partially supported by the SFB/TR 45 ‘Periods, moduli spaces, and arithmetic of algebraic varieties’. The authors would specifically like to thank N. Addington, F. Charles, J.-L. Colliot-Thélène, B. Hassett, R. Laza, M.-A. Knus, E. Macrì, R. Parimala, and V. Suresh for many helpful discussions.

1. NONTRIVIALITY CRITERIA FOR CLIFFORD INVARIANTS

In this section, by means of straightforward lattice-theoretic calculations, we describe a class of cubic fourfolds containing a plane with nontrivial Clifford invariant.

If (H, b) is a \mathbb{Z} -lattice and $A \subset H$, then the orthogonal complement $A^\perp = \{v \in H : b(v, A) = 0\}$ is a *saturated* sublattice (i.e., $A^\perp = A^\perp \otimes_{\mathbb{Z}} \mathbb{Q} \cap H$) and is thus a *primitive* sublattice (i.e., H/A^\perp is torsion free). Denote by $d(H, b) \in \mathbb{Z}$ the *discriminant*, i.e., the determinant of the Gram matrix.

Let X be a smooth cubic fourfold over \mathbb{C} . The integral Hodge conjecture holds for X (by [26], [35], cf. [34, Thm. 18]) and we denote by $A(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ the lattice of integral middle Hodge classes, which are all algebraic.

Now suppose that X contains a plane P and let $\pi : \tilde{X} \rightarrow \mathbb{P}^2$ be the quadric surface bundle defined by blowing up and projecting away from P . Let \mathcal{C}_0 be the even Clifford algebra associated to π , cf. [20] or [1, §2]. Throughout, we always assume that π has *simple degeneration*, i.e., the fibers of π have at most isolated singularities. This is equivalent to the condition that X doesn't contain another plane intersecting P ; see [32, Lemme 2]. This implies that the degeneration divisor $D \subset \mathbb{P}^2$ is a smooth sextic curve, the *discriminant cover* $f : S \rightarrow \mathbb{P}^2$ branched along D is a smooth K3 surface of degree 2, and that \mathcal{C}_0 defines an Azumaya quaternion algebra over S , cf. [20, Prop. 3.13]. We refer to the Brauer class $\beta \in \text{Br}(S)[2]$ of \mathcal{C}_0 as the *Clifford invariant* of X .

Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class associated to the embedding $X \subset \mathbb{P}^5$. The *transcendental* lattice $T(X)$, the *nonspecial cohomology* lattice K , and the *primitive cohomology* lattice $H^4(X, \mathbb{Z})_0$ are the orthogonal complements (with respect to the cup product polarization b_X) of $A(X)$, $\langle h^2, P \rangle$, and $\langle h^2 \rangle$ inside $H^4(X, \mathbb{Z})$, respectively. Thus $T(X) \subset K \subset H^4(X, \mathbb{Z})_0$. We have that $T(X) = K$ for a very general cubic fourfold, cf. the proof of [32, Prop. 2]. There are natural polarized Hodge structures on $T(X)$, K , and $H^4(X, \mathbb{Z})_0$ given by restriction from $H^4(X, \mathbb{Z})$.

Similarly, let S be a smooth integral projective surface over \mathbb{C} and $\text{NS}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ its Néron–Severi lattice. Let $h_1 \in \text{NS}(S)$ be a fixed anisotropic class. The *transcendental* lattice $T(S)$ and the *primitive cohomology* $H^2(S, \mathbb{Z})_0$ are the orthogonal complements (with respect to the cup product polarization b_S) of $\text{NS}(S)$ and $\langle h_1 \rangle$ inside $H^2(S, \mathbb{Z})$, respectively. If $f : S \rightarrow \mathbb{P}^2$ is a double cover, then we take h_1 to be the class of $f^* \mathcal{O}_{\mathbb{P}^2}(1)$.

Let $F(X)$ be the Fano variety of lines in X and $W \subset F(X)$ the divisor consisting of lines meeting P . Then W is identified with the relative Hilbert scheme of lines in the fibers of π . Its Stein factorization $W \xrightarrow{p} S \xrightarrow{f} \mathbb{P}^2$ displays W as a smooth conic bundle over the discriminant cover. Then the Abel–Jacobi map

$$\Phi : H^4(X, \mathbb{Z}) \rightarrow H^2(W, \mathbb{Z})$$

becomes an isomorphism of \mathbb{Q} -Hodge structures $\Phi : H^4(X, \mathbb{Q}) \rightarrow H^2(W, \mathbb{Q})(-1)$; see [32, Prop. 1]. Finally, $p : W \rightarrow S$ is a smooth conic bundle and there is an injective (see [33, Lemma 7.28]) morphism of Hodge structures $p^* : H^2(S, \mathbb{Z}) \rightarrow H^2(W, \mathbb{Z})$.

We recall a result of Voisin [32, Prop. 2].

Proposition 1. *Let X be a smooth cubic fourfold containing a plane. Then $\Phi(K) \subset p^* H^2(S, \mathbb{Z})_0(-1)$ is a polarized Hodge substructure of index 2.*

Proof. That $\Phi(K) \subset p^* H^2(S, \mathbb{Z})_0$ is an inclusion of index 2 is proved in [32, Prop. 2]. We now verify that the inclusion respects the Hodge filtrations. The Hodge filtration of $\Phi(K) \otimes_{\mathbb{Z}} \mathbb{C}$ is that induced from $H^2(W, \mathbb{C})(-1)$ since Φ is an isomorphism of \mathbb{Q} -Hodge structures. On the other hand, since $p : W \rightarrow S$ is a smooth conic bundle, $R^1 p_* \mathbb{C} = 0$. Hence $p^* : H^2(S, \mathbb{C}) \rightarrow H^2(W, \mathbb{C})$ is injective by the Leray spectral sequence and $p^* H^{p,2-p}(S) = p^* H^2(S, \mathbb{C}) \cap H^{p,2-p}(W)$. Thus the Hodge filtration of $p^* H^2(S, \mathbb{C})(-1)$ is induced from $H^2(W, \mathbb{C})(-1)$, and similarly for primitive cohomology. In particular, the inclusion $\Phi(K) \subset p^* H^2(S, \mathbb{Z})_0(-1)$ is a morphism of Hodge structures. Finally, by [32, Prop. 2], we have that $b_X(x, y) = -b_S(\Phi(x), \Phi(y))$ for $x, y \in K$, and thus the inclusion also preserves the polarizations. \square

By abuse of notation (of which we are already guilty), for $x \in K$, we will consider $\Phi(x)$ as an element of $p^* H^2(S, \mathbb{Z})_0(-1)$ without explicitly mentioning so.

Corollary 2. *Let X be a smooth cubic fourfold containing a plane. Then $\Phi(T(X)) \subset p^*T(S)(-1)$ is a sublattice of index ϵ dividing 2. In particular, $\text{rk } A(X) = \text{rk } \text{NS}(S) + 1$ and $d(A(X)) = 2^{2(\epsilon-1)}d(\text{NS}(S))$.*

Proof. By the saturation property, $T(X)$ and $T(S)$ coincide with the orthogonal complement of $A(X) \cap K$ in K and $\text{NS}(S) \cap H^2(S, \mathbb{Z})_0$ in $H^2(S, \mathbb{Z})_0$, respectively. Now, for $x \in T(X)$ and $a \in \text{NS}(S)_0$, we have

$$b_S(\Phi(x), a) = -\frac{1}{2}\Phi(x) \cdot g \cdot p^*a = -\frac{1}{2}b_X(x, {}^t\Phi(g \cdot p^*a)) = 0$$

by [32, Lemme 3] and the fact that ${}^t\Phi(g \cdot p^*a) \in A(X)$ (here, $g \in H^2(W, \mathbb{Z})$ is the pullback of the hyperplane class from the canonical grassmannian embedding), which follows since ${}^t\Phi : H^4(W, \mathbb{Z}) \cong H_2(W, \mathbb{Z}) \rightarrow H_4(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z})$ preserves the Hodge structure by the same argument as in the proof of Proposition 1. Therefore $\Phi(T(X)) \subset p^*T(S)(-1)$.

Since $T(X) \subset K$ and $T(S)(-1) \subset H^2(S, \mathbb{Z})_0(-1)$ are saturated (hence primitive) sublattices, an application of the snake lemma shows that $p^*T(S)(-1)/\Phi(T(X)) \subset p^*H^2(S, \mathbb{Z})_0/\Phi(K) \cong \mathbb{Z}/2\mathbb{Z}$, hence the index of $\Phi(T(X))$ in $p^*T(S)(-1)$ divides 2.

We now verify the final claims. We have $\text{rk } K = \text{rk } H^2(X, \mathbb{Z}) - 2 = \text{rk } T(X) + \text{rk } A(X) - 2$ and $\text{rk } H^2(S, \mathbb{Z})_0 = \text{rk } H^2(S, \mathbb{Z}) - 1 = \text{rk } T(S) + \text{rk } \text{NS}(S) - 1$ (since P , h^2 , and h_1 are anisotropic vectors, respectively), while $\text{rk } K = \text{rk } H^2(S, \mathbb{Z})_0$ and $\text{rk } T(X) = \text{rk } T(S)$ by Proposition 1 and the above, respectively. The claim concerning the discriminant then follows by standard lattice theory. \square

Let $Q \in A(X)$ be the class of a fiber of the quadric surface bundle $\pi : \tilde{X} \rightarrow \mathbb{P}^2$. Then $P + Q = h^2$, see [32, §1].

Proposition 3. *Let X be a smooth cubic fourfold containing a plane P . If $A(X)$ has rank 3 and even discriminant (e.g., if the associated K3 surface S of degree 2 has Picard rank 2 and even Néron–Severi discriminant) then the Clifford invariant $\beta \in \text{Br}(S)$ of X is nontrivial.*

Proof. The Clifford invariant $\beta \in \text{Br}(S)$ associated to the quadric surface bundle $\pi : \tilde{X} \rightarrow \mathbb{P}^2$ is trivial if and only if π has a rational section; see [16, Thm. 6.3] or [28, 2 Thm. 14.1, Lemma 14.2]. Such a section exists if and only if there exists an algebraic cycle $R \in A(X)$ such that $R \cdot Q = 1$; see [11, Thm. 3.1] or [21, Prop. 4.7].

Suppose that such a cycle R exists and consider the sublattice $\langle h^2, Q, R \rangle \subset A(X)$. Its intersection form has Gram matrix

$$(1) \quad \begin{array}{ccccc} & & h^2 & Q & R \\ h^2 & & 3 & 2 & x \\ Q & & 2 & 4 & 1 \\ R & & x & 1 & y \end{array}$$

for some $x, y \in \mathbb{Z}$. The determinant of this matrix is always congruent to 5 modulo 8, so this lattice cannot be a finite index sublattice of $A(X)$, which has even discriminant by hypothesis. Hence no such 2-cycle R exists and thus β is nontrivial. The final claim follows directly from Corollary 2.

Finally, if the associated K3 surface S of degree 2 has Picard rank 2 and even Néron–Severi discriminant, then $A(X)$ has rank 3 and even discriminant by Corollary 2. \square

We now provide an explicit geometric condition for the nontriviality of the Clifford invariant, which will be necessary in §4. We say that a cubic fourfold X containing a plane has a *tangent conic* if there exists a conic $C \subset \mathbb{P}^2$ everywhere tangent to the discriminant curve $D \subset \mathbb{P}^2$ of the associated quadric surface bundle.

Proposition 4. *Let X be a smooth cubic fourfold containing a plane. Let S be the associated K3 surface of degree 2 and $\beta \in \text{Br}(S)$ the Clifford invariant. If X has a tangent conic and S has Picard rank 2 then β is nontrivial.*

Proof. Consider the pull back of the cycle class of C to S via the discriminant double cover $f : S \rightarrow \mathbb{P}^2$. Then f^*C has two components C_1 and C_2 . The sublattice of the Néron–Severi lattice of S generated by $h_1 = f^*\mathcal{O}_{\mathbb{P}^2}(1) = (C_1 + C_2)/2$ and C_1 has intersection form with Gram matrix

$$\begin{array}{cc} & \begin{matrix} h_1 & C_1 \end{matrix} \\ \begin{matrix} h_1 \\ C_1 \end{matrix} & \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \end{array}$$

having determinant -8 . As S has Picard rank 2, then the entire Néron–Severi lattice is in fact generated by h_1 and C_1 (see [7, §2] for further details) and we can apply Proposition 3 to conclude the nontriviality of the Clifford invariant. \square

Remark 5. Kuznetsov’s conjecture implies that the general cubic fourfold containing a plane (i.e., the associated K3 surface S of degree 2 has Picard rank 1) is not rational. Indeed, in this case there exists no K3 surface S' with $\mathbf{A}_X \cong \mathbf{D}^b(S')$; see [21, Prop. 4.8]. Therefore, for any rational cubic fourfold containing a plane, S should have Picard rank at least 2.

2. THE CLIFFORD INVARIANT ON $\mathcal{C}_8 \cap \mathcal{C}_{14}$

In this section, we first prove that $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has five irreducible components and we describe each of them in lattice theoretic terms. We then completely analyze the (non)triviality of the Clifford invariant of the general cubic fourfold (i.e., such that $A(X)$ has rank 3) in each irreducible component. One of the components corresponds to cubic fourfolds containing a plane and having a tangent conic (i.e., those considered in Proposition 4), where we already know the nontriviality of the Clifford invariant. Another component corresponds to cubic fourfolds containing two disjoint planes, where we already know the triviality of the Clifford invariant. There are another two components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ whose general elements have trivial Clifford invariant (see Proposition 7), answering a question of F. Charles.

We recall that a cubic fourfold X is in \mathcal{C}_8 or \mathcal{C}_{14} if and only if $A(X)$ has a primitive sublattice $K_8 = \langle h^2, P \rangle$ or $K_{14} = \langle h^2, T \rangle$ having Gram matrix

$$\begin{array}{ccc} & \begin{matrix} h^2 & P \end{matrix} & \\ \begin{matrix} h^2 \\ P \end{matrix} & \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} & \end{array} \quad \text{or} \quad \begin{array}{ccc} & \begin{matrix} h^2 & T \end{matrix} & \\ \begin{matrix} h^2 \\ T \end{matrix} & \begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix} & \end{array}$$

respectively. This follows from the definition of \mathcal{C}_d , together with the fact that for any $d \not\equiv 0 \pmod{9}$ there is a unique lattice (up to isomorphism) of rank 2 that represents 3 and has discriminant d .

Thus a cubic fourfold X in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has a sublattice $\langle h^2, P, T \rangle \subset A(X)$ with Gram matrix

$$(2) \quad \begin{array}{ccc} & \begin{matrix} h^2 & P & T \end{matrix} & \\ \begin{matrix} h^2 \\ P \\ T \end{matrix} & \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & \tau \\ 4 & \tau & 10 \end{pmatrix} & \end{array}$$

for some $\tau \in \mathbb{Z}$ depending on X . There may be *a priori* restrictions on the possible values of τ .

Denote by A_τ the lattice of rank 3 whose bilinear form has Gram matrix (2). We will write $\mathcal{C}_\tau = \mathcal{C}_{A_\tau} \subset \mathcal{C}$ for the locus of smooth cubic fourfolds such that there is a primitive embedding $A_\tau \subset A(X)$ of lattices preserving h^2 . If nonempty, each \mathcal{C}_τ is a subvariety of codimension 2 by a variant of the proof of [10, Thm. 3.1.2].

We will use the work of Laza [22], Looijenga [23], and Mayanskiy [25, Thm. 6.1, Rem. 6.3] to classify exactly which values of τ are supported by cubic fourfolds.

Theorem 6. *The irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ are the subvarieties \mathcal{C}_τ for $\tau \in \{-1, 0, 1, 2, 3\}$. Moreover, the general cubic fourfold X in \mathcal{C}_τ satisfies $A(X) \cong A_\tau$.*

Proof. By construction, $\mathcal{C}_8 \cap \mathcal{C}_{14}$ is the union of \mathcal{C}_τ for all $\tau \in \mathbb{Z}$. First we decide for which values of τ is \mathcal{C}_τ possibly nonempty. If X is a smooth cubic fourfold, then $A(X)$ is positive definite by the Riemann bilinear relations. Hence, to be realized as a sublattice of some $A(X)$, the lattice

A_τ must be positive definite, which by Sylvester's criterion, is equivalent to A_τ having positive discriminant. As $d(A_\tau) = -3\tau^2 + 8\tau + 32$, the only values of τ making a positive discriminant are $\{-2, -1, 0, 1, 2, 3, 4\}$.

Then, we prove that \mathcal{C}_τ is empty for $\tau = -2, 4$ by demonstrating *roots* (i.e., primitive vectors of norm 2) in $A_{\tau,0} = \langle h^2 \rangle^\perp$ (see [32, §4 Prop. 1], [23, §2], or [22, Def. 2.16] for details on roots). Indeed, the vectors $(1, -3, 0)$ and $(0, -4, 1)$ form a basis for $A_{\tau,0} \subset A_\tau$; for $\tau = -2$, we find short roots $(-2, 2, 1)$ and $(2, -10, 1)$; for $\tau = 4$, we find short roots $\pm(1, 1, -1)$. Hence \mathcal{C}_τ is possibly nonempty only for $\tau \in \{-1, 0, 1, 2, 3\}$. The corresponding discriminants $d(A_\tau)$ are $\{21, 32, 37, 36, 29\}$.

For the remaining values of τ , we prove that \mathcal{C}_τ is nonempty. To this end, we verify conditions 1)–6) of [25, Thm. 6.1], proving that $A_\tau = A(X)$ for some cubic fourfold X . Condition 1) is true by definition. For condition 2), letting $v = (x, -3x - 4y, y) \in A_{\tau,0}$ we see that

$$(3) \quad b(v, v) = 2(12x^2 + (36 - 3\tau)xy + (29 - 4\tau)y^2)$$

is even. For condition 5), letting $w = (x, y, z) \in A_\tau$, we compute that

$$(4) \quad b(h^2, w)^2 - b(w, w) = 2(3x^2 - y^2 + z^2 + 2xy + 8xz + (4 - \tau)yz)$$

is even. For conditions 3)–4), given each of the five values of τ , we use standard Diophantine techniques to prove the nonexistence of short and long roots of (3).

Finally, for condition 6), let $q_{K_\tau} : A_\tau^*/A_\tau \rightarrow \mathbb{Q}/2\mathbb{Z}$ be the discriminant form of (4), restricted to the discriminant group A_τ^*/A_τ of the lattice A_τ . Appealing to Nikulin [27, Cor. 1.10.2], it suffices to check that the signature satisfies $\text{sgn}(q_{K_\tau}) \equiv 0 \pmod{8}$; cf. [25, Rem. 6.3]. Employing the notation of [27, Prop. 1.8.1], we compute the finite quadratic form q_{K_τ} in each case:

(5)

τ	-1	0	1	2	3
$d(A_\tau)$	21	32	37	36	29
A_τ^*/A_τ	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$	$\mathbb{Z}/37\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/29\mathbb{Z}$
q_{K_τ}	$q_1^3(3) \oplus q_1^7(7)$	$q_3^2(2) \oplus q_1^2(2^4)$	$q_\theta^{37}(37)$	$q_3^2(2) \oplus q_1^2(2) \oplus q_1^3(3^2)$	$q_\theta^{29}(29)$

where θ represents a nonsquare class modulo the respective odd prime. In each case of (5), we verify the signature condition using the formulas in [27, Prop. 1.11.2].

Finally, for the five values of τ , we prove that \mathcal{C}_τ is irreducible. As the rank of $A(X)$ is an upper-semicontinuous function on \mathcal{C} , the general cubic fourfold X in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has $A(X)$ of rank 3 (by the argument above), of which A_τ is a finite index sublattice for some τ . Each proper finite overlattice B of A_τ , such that B (along with its sublattices K_8 and K_{14}) is primitively embedded into $H^4(X, \mathbb{Z})$, will give rise to an irreducible component of \mathcal{C}_τ . We will prove that no such proper finite overlattices exist. For $\tau \in \{21, 37, 29\}$, the discriminant of A_τ is squarefree, so there are no proper finite overlattices. In the case $\tau = 0, 2$, we note that $B_0 = \langle h^2 \rangle^\perp$ is a proper finite overlattice of the binary lattice $A_{\tau,0}$ (as $\langle h^2 \rangle \subset B$ is assumed primitive). We then directly compute that each such B_0 has *long roots* (i.e., vectors of norm 6 whose pairing with any other vector is divisible by 3). Therefore, no such proper finite overlattices exist. \square

We now address the question of the (non)triviality of the Clifford invariant.

Proposition 7. *Let X be a general cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ (so that $A(X)$ has rank 3). The Clifford invariant is trivial if and only if τ is odd.*

Proof. If τ is odd then, as in the proof of Proposition 8, $(P+T).Q = -\tau$ is odd, hence the Clifford invariant $\beta \in \text{Br}(S)$ is trivial by an application of the criteria in [11, Thm. 3.1] or [21, Prop. 4.7] (cf. the proof of Proposition 3). If τ is even, then $A_d = A(X)$ has rank 3 and even discriminant, hence β is nontrivial by Proposition 3. \square

For $\tau = -1$, the component \mathcal{C}_τ consists of cubic fourfolds containing two disjoint planes (see [10, 4.1.3]). We now give a geometric description of the general member of \mathcal{C}_τ for $\tau = 0$.

Proposition 8. *Let X be a smooth cubic fourfold containing a plane P and having a tangent conic such that $A(X)$ has rank 3. Then X is in the component \mathcal{C}_τ for $\tau = 0$.*

Proof. Since X has a tangent conic and $A(X)$ has rank 3, $A(X)$ has discriminant 8 or 32 and X has nontrivial Clifford invariant by Corollary 2 and Proposition 4. As the sublattice $\langle h^2, P \rangle \subset A(X)$ is primitive, we can choose a class $T \in A(X)$ such that $\langle h^2, P, T \rangle \subset A(X)$ has discriminant 32. Adjusting T by a multiple of P , we can assume that $h^2.T = 4$. Write $\tau = P.T$.

Adjusting T by multiples of $h^2 - 3P$ keeps $h^2.T = 4$ and adjusts τ by multiples of 8. The discriminant being 32, we are left with two possible choices $(\tau = 0, 4)$ for the Gram matrix of $\langle h^2, P, T \rangle$ up to isomorphism:

$$\begin{array}{ccc|ccc} & h^2 & P & T & & h^2 & P & T \\ h^2 & 3 & 1 & 4 & & h^2 & 3 & 1 & 4 \\ P & 1 & 3 & 0 & & P & 1 & 3 & 4 \\ T & 4 & 0 & 10 & & T & 4 & 4 & 12 \end{array}$$

In these cases, we compute that $K \cap \langle h^2, P, T \rangle$ (i.e., the orthogonal complement of $\langle h^2, P \rangle$ in $\langle h^2, P, T \rangle$) is generated by $3h^2 - P - 2T$ and $h^2 + P - T$ and has discriminant 16 and 5, respectively. Let S be the associated K3 surface of degree 2. We calculate that $\text{NS}(S) \cap H^2(S, \mathbb{Z})_0$ (i.e., the orthogonal complement of $\langle h_1 \rangle$ in $\text{NS}(S)$) is generated by $h_1 - C_1$ and has discriminant -4 (see Proposition 4 for definitions). Arguing as in the proof of Corollary 2, there is a lattice inclusion $\Phi(K \cap \langle h^2, P, T \rangle) \subset \text{NS}(S) \cap H^2(S, \mathbb{Z})_0(-1)$ having index dividing 2, which rules out the second case above by comparing discriminants. \square

In Proposition 7 we isolate three classes of smooth cubic fourfolds $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ with *trivial* Clifford invariant. In particular, such cubic fourfolds are rational and verify Kuznetsov's conjecture; see [21, Prop. 4.7]. While the component \mathcal{C}_τ for $\tau = -1$ is in the complement of the pfaffian locus (see [30, Prop. 1b]), we expect that the pfaffian locus is dense in the other four components.

3. THE TWISTED DERIVED EQUIVALENCE

Homological projective duality (HPD) can be used to obtain a significant semiorthogonal decomposition of the derived category of a pfaffian cubic fourfold. As the universal pfaffian variety is singular, a noncommutative resolution of singularities is required to establish HPD in this case. A *noncommutative resolution of singularities* of a scheme Y is a coherent \mathcal{O}_Y -algebra \mathcal{R} with finite homological dimension that is generically a matrix algebra (these properties translate to “smoothness” and “birational to Y ” from the categorical language). We refer to [18] for details on HPD.

Theorem 9 ([17]). *Let W be a \mathbb{C} -vector space of dimension 6 and $Y \subset \mathbb{P}(\wedge^2 W^\vee)$ the universal pfaffian cubic hypersurface. There exists a noncommutative resolution of singularities (Y, \mathcal{R}) that is HP dual to the grassmannian $\text{Gr}(2, W)$. In particular, the bounded derived category of a smooth pfaffian cubic fourfold X admits a semiorthogonal decomposition*

$$\text{D}^b(X) = \langle \text{D}^b(S'), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle,$$

where S' is a smooth K3 surface of degree 14. In particular, $\mathbf{A}_X \cong \text{D}^b(S')$.

Proof. The relevant noncommutative resolution of singularities \mathcal{R} of Y is constructed in [19]. The HP duality is established in [17, Thm. 1]. The semiorthogonal decomposition is constructed as follows. Any pfaffian cubic fourfold X is an intersection of $Y \subset \mathbb{P}(\wedge^2 W^\vee) = \mathbb{P}^{14}$ with a linear subspace $\mathbb{P}^5 \subset \mathbb{P}^{14}$. If X is smooth, then $\mathcal{R}|_X$ is Morita equivalent to \mathcal{O}_X . Via classical projective duality, $Y \subset \mathbb{P}^{14}$ corresponds to $\mathbb{G}(2, W) \subset \check{\mathbb{P}}^{14}$ while $\mathbb{P}^5 \subset \mathbb{P}^{14}$ corresponds to a linear subspace $\mathbb{P}^8 \subset \check{\mathbb{P}}^{14}$. The intersection of $\mathbb{G}(2, W)$ and \mathbb{P}^8 inside $\check{\mathbb{P}}^{14}$ is a K3 surface S' of degree 14. Kuznetsov [17, Thm. 2] describes a semiorthogonal decomposition

$$\text{D}^b(X) = \langle \mathcal{O}_X(-3), \mathcal{O}_X(-2), \mathcal{O}_X(-1), \text{D}^b(S') \rangle.$$

To obtain the desired semiorthogonal decomposition and the equivalence $\mathbf{A}_X \cong \text{D}^b(S')$, we act on $\text{D}^b(X)$ by the autoequivalence $-\otimes \mathcal{O}_X(3)$, then mutate the image of $\text{D}^b(S')$ to the left with respect to its left orthogonal complement; see [5]. This displays the left orthogonal complement of $\langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$, which is \mathbf{A}_X by definition, as a category equivalent to $\text{D}^b(S')$. \square

Finally, assuming the result in §4, we can give a proof of Theorem B.

Proof of Theorem B. Let X be a smooth complex pfaffian cubic fourfold containing a plane, S the associated K3 surface of degree 2, $\beta \in \text{Br}(S)$ the Clifford invariant, and S' the K3 surface of degree 14 arising from Theorem 9 via projective duality. Then by [21, Thm. 4.3] and Theorem 9, the category \mathbf{A}_X is equivalent to both $D^b(S, \beta)$ and $D^b(S')$.

The cubic fourfold X is rational, being pfaffian (see [4, Prop. 5ii], [29], and [3, Prop. 9.2a]). The existence of such cubic fourfolds with β nontrivial is guaranteed by Theorem 11. Thus there is a twisted derived equivalence $D^b(S, \beta) \cong D^b(S')$ between K3 surfaces of degree 2 and 14. \square

Remark 10. By [14, Rem. 7.10], given any K3 surface S and any nontrivial $\beta \in \text{Br}(S)$, there is no equivalence between $D^b(S, \beta)$ and $D^b(S)$. Thus any X as in Theorem B validates Kuznetsov's conjecture on the rationality of cubic fourfolds containing a plane, but not via the K3 surface S .

4. A PFAFFIAN CONTAINING A PLANE

In this section, we exhibit a smooth pfaffian cubic fourfold X containing a plane and having a tangent conic such that $A(X)$ has rank 3. By Propositions 3 and 8, X has nontrivial Clifford invariant and is in the $\tau = 0$ component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$. In particular, this proves that the pfaffian locus nontrivially intersects, and hence is dense in (since it is open in \mathcal{C}_{14}), the component \mathcal{C}_τ with $\tau = 0$.

Theorem 11. *Let A be the 6×6 antisymmetric matrix*

$$\begin{pmatrix} 0 & y+u & x+y+u & u & z & y+u+v \\ & 0 & x+y+z & x+z+u+w & y+z+u+v+w & x+y+z+u+v+w \\ & & 0 & x+y+u+w & x+y+u+v+w & x+y+z+v+w \\ & & & 0 & x+u+v+w & x+u+w \\ & & & & 0 & z+u+w \\ & & & & & 0 \end{pmatrix}$$

of linear forms in $\mathbb{Q}[x, y, z, u, v, w]$ and let $X \subset \mathbb{P}^5$ be the cubic fourfold defined by the vanishing of the pfaffian of A :

$$\begin{aligned} & (x - 4y - z)u^2 + (-x - 3y)uv + (x - 3y)uw + (x - 2y - z)vw - 2yv^2 + xw^2 \\ & + (2x^2 + xz - 4y^2 + 2z^2)u + (x^2 - xy - 3y^2 + yz - z^2)v + (2x^2 + xy + 3xz - 3y^2 + yz)w \\ & + x^3 + x^2y + 2x^2z - xy^2 + xz^2 - y^3 + yz^2 - z^3. \end{aligned}$$

Then:

- a) X is smooth, rational, and contains the plane $P = \{x = y = z = 0\}$.
- b) The degeneration divisor $D \subset \mathbb{P}^2$ of the associated quadric surface bundle $\pi : \tilde{X} \rightarrow \mathbb{P}^2$ is the sextic curve given by the vanishing of:

$$\begin{aligned} d = & x^6 + 6x^5y + 12x^5z + x^4y^2 + 22x^4yz + 28x^3y^3 - 38x^3y^2z + 46x^3yz^2 + 4x^3z^3 \\ & + 24x^2y^4 - 4x^2y^3z - 37x^2y^2z^2 - 36x^2yz^3 - 4x^2z^4 + 48xy^4z - 24xy^3z^2 \\ & + 34xy^2z^3 + 4xyz^4 + 20y^5z + 20y^4z^2 - 8y^3z^3 - 11y^2z^4 - 4yz^5. \end{aligned}$$

This curve is smooth; in particular, π has simple degeneration and the discriminant cover is a smooth K3 surface S of degree 2.

- c) The conic $C \subset \mathbb{P}^2$ defined by the vanishing of $x^2 + yz$ is tangent to the degeneration divisor D at six points (five of which are distinct).
- d) The K3 surface S has (geometric) Picard rank 2.

In particular, the Clifford invariant of X is geometrically nontrivial.

Proof. Verifying smoothness of X and D is a straightforward application of the jacobian criterion, while the inclusion $P \subset X$ is checked by inspecting the expression for $\text{pf}(A)$; every monomial is divisible by x , y or z . Rationality comes from being a pfaffian cubic fourfold; see [29]. The smoothness of D implies that π has simple degeneration; see [13, Rem. 7.1] or [1, Rem. 2.6]. This establishes parts *a*) and *b*).

For part *c*), note that we can write the equation for the degeneration divisor as $d = (x^2 + yz)f + g^2$, where

$$\begin{aligned} f &= x^4 + 6x^3y + 12x^3z + x^2y^2 + 21x^2yz - 25x^2z^2 + 28xy^3 \\ &\quad - 24xy^2z + 34xyz^2 + 4xz^3 + 20y^4 - 5y^3z - 8y^2z^2 - 11yz^3 - 4z^4. \\ g &= 2xy^2 + 5y^2z - 5x^2z. \end{aligned}$$

Hence the conic $C \subset \mathbb{P}^2$ defined by $x^2 + yz$ is tangent to D along the zero-dimensional scheme of length 6 given by the intersection of C and the vanishing of g .

For part *d*), the surface S is the smooth sextic in $\mathbb{P}(1, 1, 1, 3) = \text{Proj } \mathbb{Q}[x, y, z, w]$ given by

$$w^2 = d(x, y, z),$$

which is the double cover \mathbb{P}^2 branched along the discriminant divisor D . In these coordinates, the discriminant cover $f : S \rightarrow \mathbb{P}^2$ is simply the restriction to S of the projection $\mathbb{P}(1, 1, 1, 3) \dashrightarrow \mathbb{P}^2$ away from the hyperplane $\{w = 0\}$. Let $C \subset \mathbb{P}^2$ be the conic from part *d*). As discussed in Proposition 4, the curve f^*C consists of two (-2) -curves C_1 and C_2 . These curves generate a sublattice of $\text{NS}(S)$ of rank 2. Hence $\rho(\overline{S}) \geq \rho(S) \geq 2$, where $\overline{S} = S \times_{\mathbb{Q}} \mathbb{C}$.

We show next that $\rho(\overline{S}) \leq 2$. Write S_p for the reduction mod p of S and $\overline{S}_p = S_p \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. Let $\ell \neq 3$ be a prime and write $\phi(t)$ for the characteristic polynomial of the action of absolute Frobenius on $H_{\text{ét}}^2(\overline{S}_3, \mathbb{Q}_{\ell})$. Then $\rho(\overline{S}_3)$ is bounded above by the number of roots of $\phi(t)$ that are of the form 3ζ , where ζ is a root of unity [31, Prop. 2.3]. Combining the Lefschetz trace formula with Newton's identities and the functional equation that $\phi(t)$ satisfies, it is possible calculate $\phi(t)$ from knowledge of $\#S(\mathbb{F}_{3^n})$ for $1 \leq n \leq 11$; see [31] for details.

Let $\tilde{\phi}(t) = 3^{-22}\phi(3t)$, so that the number of roots of $\tilde{\phi}(t)$ that are roots of unity gives an upper bound for $\rho(\overline{S}_3)$. Using *Magma*, we compute

$$\tilde{\phi}(t) = \frac{1}{3}(t-1)^2(3t^{20} + t^{19} + t^{17} + t^{16} + 2t^{15} + 3t^{14} + t^{12} + 3t^{11} + 2t^{10} + 3t^9 + t^8 + 3t^6 + 2t^5 + t^4 + t^3 + t + 3)$$

The roots of the degree 20 factor of $\tilde{\phi}(t)$ are not integral, and hence they are not roots of unity. We conclude that $\rho(\overline{S}_3) \leq 2$. By [31], we have $\rho(\overline{S}) \leq \rho(\overline{S}_3)$, so $\rho(\overline{S}) \leq 2$. It follows that S (and \overline{S}) has Picard rank 2. This concludes the proof of part *d*).

Finally, the nontriviality of the Clifford invariant follows from Propositions 3 and 4. \square

A satisfying feature of Theorem 11 is that we can write out a representative of the Clifford invariant of X explicitly, as a quaternion algebra over the function field of the K3 surface S . We first prove a handy lemma, of independent interest for its arithmetic applications (see e.g., [12, 13]).

Lemma 12. *Let K be a field of characteristic $\neq 2$ and q a nondegenerate quadratic form of rank 4 over K with discriminant extension L/K . For $1 \leq r \leq 4$ denote by m_r the determinant of the leading principal $r \times r$ minor of the symmetric Gram matrix of q . Then the class $\beta \in \text{Br}(L)$ of the even Clifford algebra of q is the quaternion algebra $(-m_2, -m_1m_3)$.*

Proof. On $n \times n$ matrices M over K , symmetric gaussian elimination is the following operation:

$$M = \begin{pmatrix} a & v^t \\ v & A \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & A - a^{-1}vv^t \end{pmatrix}$$

where $a \in K^{\times}$, $v \in K^{n-1}$ is a column vector, and A is an $(n-1) \times (n-1)$ matrix over K . Then $m_1 = a$ and the element in the first row and column of $A - a^{-1}vv^t$ is precisely m_2/m_1 . By induction,

M can be diagonalized, using symmetric gaussian elimination, to the matrix

$$\text{diag}(m_1, m_2/m_1, \dots, m_n/m_{n-1}).$$

For q of rank 4 with symmetric Gram matrix M , we have

$$q = \langle m_1 \rangle \otimes \langle 1, m_2, m_1 m_2 m_3, m_1 m_3 m_4 \rangle$$

so that over $L = K(\sqrt{m_4})$, we have that $q \otimes_K L = \langle m_1 \rangle \otimes \langle 1, m_2, m_1 m_3, m_1 m_2 m_3 \rangle$, which is similar to the norm form of the quaternion L -algebra with symbol $(-m_2, -m_1 m_3)$. Thus the even Clifford algebra of q is Brauer equivalent to $(-m_2, -m_1 m_3)$ over L . \square

Proposition 13. *The Clifford invariant of the fourfold X of Theorem 11 is represented by the unramified quaternion algebra (b, ac) over the function field of associated K3 surface S , where*

$$a = x - 4y - z, \quad b = x^2 + 14xy - 23y^2 - 8yz,$$

and

$$c = 3x^3 + 2x^2y - 4x^2z + 8xyz + 3xz^2 - 16y^3 - 11y^2z - 8yz^2 - z^3.$$

Proof. The symmetric Gram matrix of the quadratic form $(\mathcal{O}_{\mathbb{P}^2}^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-1), q, \mathcal{O}_{\mathbb{P}^2}(1))$ of rank 4 over \mathbb{P}^2 associated to the quadric bundle $\pi : \tilde{X} \rightarrow \mathbb{P}^2$ is

$$\begin{pmatrix} 2(x - 4y - z) & -x - 3y & x - 3y & 2x^2 + xz - 4y^2 + 2z^2 \\ & 2(-2y) & x - 2y - z & x^2 - xy - 3y^2 + yz - z^2 \\ & & 2x & 2x^2 + xy + 3xz - 3y^2 + yz \\ & & & 2(x^3 + x^2y + 2x^2z - xy^2 + xz^2 - y^3 + yz^2 - z^3) \end{pmatrix}$$

see [13, §4.2] or [21, §4]. Since S is regular, $\text{Br}(S) \rightarrow \text{Br}(k(S))$ is injective; see [2] or [9, Cor. 1.10]. By functoriality of the Clifford algebra, the generic fiber $\beta \otimes_S k(S) \in \text{Br}(k(S))$ is represented by the even Clifford algebra of the generic fiber $q \otimes_{\mathbb{P}^2} k(\mathbb{P}^2)$. Thus we can perform our calculations in the function field $k(S)$. In the notation of Lemma 12, we have $m_1 = 2a$, $m_2 = -b$, and $m_3 = -2c$, and the formulas follow immediately. \square

Remark 14. Contrary to the situation in [13], the transcendental Brauer class $\beta \in \text{Br}(S)$ is *constant* when evaluated on $S(\mathbb{Q})$; this suggests that arithmetic invariants do not suffice to witness the non-triviality of β . Indeed, using elimination theory, we find that the odd primes p of bad reduction of S are 5, 23, 263, 509, 1117, 6691, 3342589, 197362715625311, and 4027093318108984867401313726363. For each odd prime p of bad reduction, we compute that the singular locus of \bar{S}_p consists of a single ordinary double point. Thus by [12, Prop. 4.1, Lemma 4.2], the local invariant map associated to β is constant on $S(\mathbb{Q}_p)$, for all odd primes p of bad reduction. By an adaptation of [12, Lemma 4.4], the local invariant map is also constant for odd primes of good reduction.

At the real place, we prove that $S(\mathbb{R})$ is connected, hence the local invariant map is constant. To this end, recall that the set of real points of a smooth hypersurface of even degree in $\mathbb{P}^2(\mathbb{R})$ consists of a disjoint union of *ovals* (i.e., topological circles, each of whose complement is homeomorphic to a union of a disk and a Möbius band, in the language of real algebraic geometry). In particular, $\mathbb{P}^2(\mathbb{R}) \setminus D(\mathbb{R})$ has a unique nonorientable connected component R . By graphing an affine chart of $D(\mathbb{R})$, we find that the point $(1 : 0 : 0)$ is contained in R . We compute that the map projecting from $(1 : 0 : 0)$ has four real critical values, hence $D(\mathbb{R})$ consists of two ovals. These ovals are not nested, as can be seen by inspecting the graph of $D(\mathbb{R})$ in an affine chart. The Gram matrix of the quadratic form, specialized at $(1 : 0 : 0)$, has positive determinant, hence by local constancy, the equation for D is positive over the entire component R and negative over the interiors of the two ovals (since D is smooth). In particular, the map $f : S(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ has empty fibers over the interiors of the two ovals and nonempty fibers over $R \subset \mathbb{P}^2(\mathbb{R})$ where it restricts to a nonsplit unramified cover of degree 2, which must be the orientation double cover of R since $S(\mathbb{R})$ is orientable (the Kähler form on S defines an orientation). In particular, $S(\mathbb{R})$ is connected.

This shows that β is constant on $S(\mathbb{Q})$. We believe that the local invariant map is also constant at the prime 2, though this must be checked with a brute force computation.

REFERENCES

- [1] A. Auel, M. Bernardara, and M. Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories and rationality problems*, [arXiv:1109.6938v1](#), 2011.
- [2] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.
- [3] A. Beauville, *Determinantal hypersurfaces*, Dedicated to William Fulton on the occasion of his 60th birthday, Michigan Math. J. **48** (2000), 39–64.
- [4] A. Beauville and R. Donagi, *La variété des droites d’une hypersurface cubique de dimension 4*, C.R. Acad. Sc. Paris, Série I **301** (1985), 703–706.
- [5] A. Bondal, *Representations of associative algebras and coherent sheaves*, Math. USSR-Izv. **34** (1990), no. 1, 23–42.
- [6] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, Computational Algebra and Number Theory, London, 1993, J. Symbolic Comput. **24** (1997), nos. 3–4, 235–265.
- [7] A.-S. Elsenhans and J. Jahnel, *K3 surfaces of Picard rank one which are double covers of the projective plane*, Higher-dimensional geometry over finite fields, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur. **16** (2008), 63–77.
- [8] G. Fano, *Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali di quarto ordine*, Comment. Math. Helv. **15** (1943), 71–80.
- [9] A. Grothendieck, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 67–87.
- [10] B. Hassett *Special cubic fourfolds*, Compos. Math. **120** (2000), no. 1, 1–23.
- [11] ———, *Some rational cubic fourfolds*, J. Algebraic Geometry **8** (1999), no. 1, 103–114.
- [12] B. Hassett, A. Várilly-Alvarado, *Failure of the Hasse principle on general K3 surfaces*, preprint [arXiv:1110.1738v1](#), 2011.
- [13] B. Hassett, A. Várilly-Alvarado, P. Várilly, *Transcendental obstructions to weak approximation on general K3 surfaces*, Adv. in Math. **228** (2011), 1377–1404.
- [14] D. Huybrechts and P. Stellari, *Equivalences of twisted K3 surfaces*, Math. Ann. **332** (2005), no. 4, 901–936.
- [15] M.-A. Knus, *Quadratic and hermitian forms over rings*, Springer-Verlag, Berlin, 1991.
- [16] M.-A. Knus, R. Parimala, and R. Sridharan, *On rank 4 quadratic spaces with given Arf and Witt invariants*, Math. Ann. **274** (1986), no. 2, 181–198.
- [17] A. Kuznetsov, *Homological projective duality for Grassmannians of lines*, preprint [arXiv:math/0610957](#), 2006.
- [18] ———, *Homological projective duality*, Publ. Math. Inst. Hautes Études Sci. (2007), no. 105, 157–220.
- [19] ———, *Lefschetz decompositions and categorical resolutions of singularities*, Sel. Math., New Ser. **13** (2007), no. 4, 661–696.
- [20] ———, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. **218** (2008), no. 5, 1340–1369.
- [21] ———, *Derived categories of cubic fourfolds*, Cohomological and geometric approaches to rationality problems, Progr. Math., vol. 282, Birkhäuser Boston Inc., Boston, MA, 2010, pp. 219–243.
- [22] R. Laza, *The moduli space of cubic fourfolds via the period map*, Ann. of Math. **172** (2010), no. 1, 673–711.
- [23] E. Looijenga, *The period map for cubic fourfolds*, Invent. Math. **177** (2009), 213–233.
- [24] E. Macri and P. Stellari, *Fano varieties of cubic fourfolds containing a plane*, preprint [arXiv:0909.2725v1](#), 2009.
- [25] E. Mayanskiy, *Intersection lattices of cubic fourfolds* preprint [arXiv:1112.0806](#), 2011.
- [26] J.P. Murre, *On the Hodge conjecture for unirational fourfolds*, Nederl. Akad. Wetensch. Proc. Ser. A **80** (Indag. Math. **39**) (1977), no. 3, 230–232.
- [27] V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Math. USSR Izv. **14** (1979), 103–167.
- [28] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften **270**, Springer-Verlag, Berlin, 1985.
- [29] S.L. Tregub, *Three constructions of rationality of a cubic fourfold*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1984), no. 3, 8–14. Translation in Moscow Univ. Math. Bull. **39** (1984), no. 3, 8–16.
- [30] ———, *Two remarks on four-dimensional cubics*, Uspekhi Mat. Nauk **48** (1993), no. 2(290), 201–202. Translation in Russian Math. Surveys **48** (1993), no. 2, 206–208.
- [31] R. van Luijk, *K3 surfaces with Picard number one and infinitely many rational points*, Algebra Number Theory **1**, (2007), no. 1, 1–15.
- [32] C. Voisin, *Théorème de Torelli pour les cubiques de \mathbb{P}^5* , Invent. Math. **86** (1986), no. 3, 577–601.
- [33] ———, *Hodge theory and complex algebraic geometry. I*. Translated from the French by Leila Schneps. Reprint of the 2002 English edition. Cambridge Studies in Advanced Mathematics **76**, Cambridge University Press, Cambridge, 2007.
- [34] ———, *Some aspects of the Hodge conjecture*, Jpn. J. Math. **2** (2007), no. 2, 261–296.
- [35] S. Zucker, *The Hodge conjecture for cubic fourfolds*, Compositio Math. **34** (1977), no. 2, 199–209.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, NY 10012, USA

E-mail address: auel@cims.nyu.edu

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, 31062 TOULOUSE CEDEX 9, FRANCE

E-mail address: marcello.bernardara@math.univ-toulouse.fr

INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES, UNIVERSITÉ DE RENNES 1, 263 AVENUE DU GÉNÉRAL LECLERC, CS 74205, 35042 RENNES CEDEX, FRANCE

E-mail address: michele.bolognesi@univ-rennes1.fr

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY MS 136, 6100 S. MAIN ST., HOUSTON, TX 77005, USA

E-mail address: av15@rice.edu